Project Report

On project done in Summer of 2018

Submitted By

Manas Vishal (16MS205)

BS-MS Dual Degree

Department of Physical Sciences

Indian Institute of Science Education and Research,
Kolkata

Supervisor

Dr. Subhadip Mitra

International Institute of Information Technology,
Hyderabad

Acknowledgement

I wish to thank Dr.Subhadip Mitra for being my guide in my project work. It was his encouragement and suggestions at each step of work which enabled me to gain conceptual knowledge of the subject.I am also grateful to the members of IIIT-Hyderabad for their kind cooperation.
I also thank my parents for their constant support throughout the project.

Manas Vishal

Contents

1 Introduction to General Relativity

It is a theory that Albert Einstein developed starting from 1907 in order to incorporate the graviational field within the framework of special relativity, and that his author presented in its final version to the Prussian Academy of Science on 25th November 1915 in the paper The Field Equations of Gravitation. In much the same way as the theory of special relativity was grounded on two postulates - the one of relativity and the one of invariant light speed in vacuum, Einstein based his theory of general relativity on two fundamental postulates:

1. *The Equivalence Principle which, by posing the equivalence of gravity and acceleration, a well known principle which was first tested by Galileo by comparing the periods of pendula with weights made out of different materials, allows to locally 'turn off' the effect of gravity.*
2. *The Principle of Covariance, which extends the principle of relativity to say that the form of the laws of physics should be the same in all - inertial and accelerating - frames*

2 Mathematical Tools Needed

For this project and to learn General Relativity, following mathematical tools are needed apart from secondary school mathematics:

2.1 Tensors and Manifolds

2.1.1 Tensors

In three dimensions, a vector $\vec{\boldsymbol{A}}$ has three components, which we refer to as A^i taking the values 1,2, or 3. The dot product of two vectors is then

$$\vec{A}.\vec{B} = \sum_{i=1}^{3} A^i B^i \equiv A^i B^i$$

where I have introduced the Einstein summation convention of not explicitly writing the $\sum$ sign when an index (in this case i) appears twice. Similarly, matrices can be written in component notation. For example, the product of two matrices M and N is

$$(\mathbf{MN})_{ij} = M_{ik}N_{kj}$$

again with implicit summation over k.

In relativity, two generalizations must be made. First, in relativity a vector has a fourth component, the time component. Since the spatial indices run from 1 to 3, it is conventional to use 0 for the time component. Greek letters are used to represent all four components, so $A^\mu = (A^0, A^i)$. The second, more subtle, feature of relativity is the distinction between upper and lower indices, the former associated with vectors and the latter with 1-forms. One goes back and forth with the metric tensor, so

$$A_\mu = g_{\mu\nu}A^\mu$$
$$A^\mu = g^{\mu\nu}A_\nu$$

A vector and a 1-form can be contracted to produce an invariant, a scalar. For example, the statement that the four-momentum squared of a massless particle must vanish is

$$P^2 \equiv P_\mu P^\mu = g_{\mu\nu}P^\mu P^\nu = 0$$

Just as the metric can turn an upper index on a vector into a lower index, the metric can be used to raise and lower indices on tensors with an arbitrary number of indices. For example, raising the indices on the metric tensor itself leads to

$$g^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}g_{\alpha\beta}$$

If the index $\alpha = \nu$, then the first term on the right is equal to the term on the left, so if the combination of the last two terms on the right force α to be equal to ν, then the equation is satisfied.Therefore,

$$g^{\nu\beta}g_{\alpha\beta} = \delta^\nu_\alpha$$

where δ^ν_α is the Kronecker delta equal to zero unless $\nu = \alpha$ in which case it is 1.

If a tensor transform according to

$$\mathbf{D^i} = \frac{\partial \mathrm{X}^{\mathbf{i}}}{\partial \mathrm{x}^{\mathbf{1}}}\mathbf{d^1} + \frac{\partial \mathrm{X}^{\mathbf{i}}}{\partial \mathrm{x}^{\mathbf{2}}}\mathbf{d^2} + \ldots + \frac{\partial \mathrm{X}^{\mathbf{i}}}{\partial \mathrm{x}^{\mathbf{n}}}\mathbf{d^n}$$

then the tensor is called **contravariant tensor**. And if it transforms according to

$$\mathbf{G^i} = \frac{\partial \mathrm{x}^{\mathbf{i}}}{\partial \mathbf{X^1}}\mathbf{g^1} + \frac{\partial \mathrm{x}^{\mathbf{i}}}{\partial \mathbf{X^2}}\mathbf{g^2} + \ldots + \frac{\partial \mathrm{x}^{\mathbf{i}}}{\partial \mathbf{X^n}}\mathbf{g^n}$$

then the tensor is called **covariant tensors**.

2.1.2 Manifolds

A manifold in a three dimensional space can be imagined as a surface. More generally, a manifold embedded in n-dimensional euclidean space locally looks like a (n-1) dimensional vector space. For example Earth (a big sphere) is a big manifold embedded in 3-dimensional space. But, we as tiny entities living on its surface can only see flat 2-dimensional land. So locally at every point on a sphere, it looks like a 2-dimensional plane. Curves in 3-dimensional space are also manifolds. But, they locally look like 1-dimensional vector space(a line). To be a little more precise, a manifold embedded in n-dimensional euclidean space locally looks like k-dimensional vector space (k<n) at every point on it.

Affine connection is a geometric object on a smooth manifold which connects nearby tangent spaces, so it permits tangent vector fields to be differentiated as if they were functions on the manifold with values in a fixed vector space.

Riemann Manifold is a real, smooth manifold M equipped with an inner product g_p on the tangent space T_pM at each point p that varies smoothly from point to point in the sense that if X and Y are differentiable vector fields on M, then $p \mapsto g_p(X|_p, Y|_p)$ is a smooth function. The family g_p of inner products is called a Riemannian metric (or Riemannian metric tensor). These terms are named after the German mathematician Bernhard Riemann. The study of Riemannian manifolds constitutes the subject called Riemannian geometry.

2.2 Covariant and Contravariant derivates / Parallel Transport

2.2.1 Derivatives

In curved space the derivatives can be represented by covariant and contravariant derivatives for tensors.

The covariant derivative of a contravariant tensor $\boldsymbol{A^a}$ (also called the "semicolon derivative" since its symbol is a semicolon) is given by

$$\begin{aligned} \boldsymbol{A}^a_{;b} &= \frac{\partial \boldsymbol{A}^a}{\partial x^b} + \Gamma^a_{bk}\boldsymbol{A}^k \\ &= \boldsymbol{A}^a_b + \Gamma^a_{bk}\boldsymbol{A}^k \end{aligned}$$

where Γ^i_{jk} is a Christoffel symbol, Einstein summation has been used in the last term, and $\boldsymbol{A}^k_{,k}$ is a comma derivative. The notation $\nabla.\boldsymbol{A}$, which is a generalization of the symbol commonly used to denote the divergence of a vector function in three dimensions, is sometimes also used.

The covariant derivative of a covariant tensor A_a is

$$\boldsymbol{A_{a;b} = \frac{\partial A_a}{\partial x^b} - \Gamma^k_{ab} A_k}$$

2.2.2 Parallel Transport

In geometry, parallel transport is a way of transporting geometrical data along smooth curves in a manifold. If the manifold is equipped with an affine connection (a covariant derivative or connection on the tangent bundle), then this connection allows one to transport vectors of the manifold along curves so that they stay parallel with respect to the connection.

The parallel transport for a connection thus supplies a way of, in some sense, moving the local geometry of a manifold along a curve: that is, of connecting the geometries of nearby points. There may be many notions of parallel transport available, but a specification of one — one way of connecting up the geometries of points on a curve — is tantamount to providing a connection. In fact, the usual notion of connection is the infinitesimal analog of parallel transport. Or, vice versa, parallel transport is the local realization of a connection.

3 General Relativity

Geodesic equation derivation:

In differential geometry, a geodesic can be simply defined as a line on a curved space. In general relativity, any body falling under the influence of gravity follows a geodesic. We can express the proper time along a time-like worldline (while ignoring the limits) as:

$$\tau = \int_w ds = \int_w \frac{ds}{d\lambda} d\lambda$$

with λ being the affine paramter parametrizing the path.

For a given proper spacetime interval $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, we get

$$\tau = \int_w \sqrt{\frac{g_{\mu\nu} dx^\mu dx^\nu}{d\lambda d\lambda}} d\lambda$$

Affine Geodesics

Absolute derivative $\frac{\mathbf{D}A^\mu}{\mathbf{D\lambda}} = (\nabla_\nu A^\mu)\frac{\mathrm{dx}^\nu}{\mathrm{d}\lambda}$

$\frac{\partial A^\mu}{\partial x^\nu} = A^\mu_{,\nu}$ and $\nabla_\nu A^\mu = A^\mu_{;\nu}$

$$\frac{\mathbf{D}A^\mu}{\mathbf{D\lambda}} = (\frac{\partial A^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\sigma} A^\sigma)\frac{\mathrm{dx}^\nu}{\mathrm{d}\lambda}$$

A vector is said to be parallely transported if

$$\frac{\mathbf{D}A^\mu}{\mathbf{D\lambda}} = 0$$

.

In general, $T^{\mu\nu}$ is parallel transported along C, if

$$\frac{\mathbf{D}T^{\mu\nu}}{\mathbf{D\lambda}} = 0$$

C is $x = x^\mu(\lambda)$.

Affine geodesics is a self parallel curve. The tangent vector to a curve C

$$\mathsf{X}^\mu = \mathsf{X}^\mu(\lambda)$$

is $\frac{d\mathsf{X}^\mu}{d\lambda}$

The vector along C is just

$$\frac{\mathbf{D}(\frac{\mathbf{dx}^\mu}{\mathbf{d\lambda}})}{\mathbf{D}\lambda} = \mathbf{0}$$

. This equation describe an affine geodesic.

$$(\frac{dx^\mu}{d\lambda})_{;\nu}(\frac{dx^\nu}{d\lambda}) = 0$$

$$[(\frac{dx^\mu}{d\lambda})_{,\nu} + \Gamma^\mu_{\nu\sigma}\frac{\mathrm{dx}^\sigma}{\mathrm{d}\lambda}](\frac{dx^\nu}{d\lambda}) = 0$$

$$(\frac{dx^\mu}{d\lambda})_{,\nu}(\frac{dx^\nu}{d\lambda}) + \Gamma^\mu_{\nu\sigma}\frac{\mathrm{dx}^\sigma}{\mathrm{d}\lambda}\frac{\mathrm{dx}^\nu}{\mathrm{d}\lambda} = 0$$

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\sigma}\frac{\mathrm{dx}^\sigma}{\mathrm{d}\lambda}\frac{\mathrm{dx}^\nu}{\mathrm{d}\lambda} = 0$$

$\frac{dx^\sigma}{d\lambda}\frac{dx^\nu}{d\lambda}$ is symmetric in (ν, σ)

Riemann and Ricci Tensors

Consider

$$\nabla_\mu \mathbf{A}^\sigma = \partial_\mu \mathbf{A}^\nu + \mathbf{\Gamma}^\nu_{\mu\alpha}\mathbf{A}^\alpha$$

so

$$\begin{aligned}\nabla_\sigma \nabla_\mu \mathbf{A}^\nu = & [\partial_\sigma \partial_\mu \mathbf{A}^\nu + \partial_\sigma \mathbf{\Gamma}^\nu_{\mu\gamma}\mathbf{A}^\gamma + \mathbf{\Gamma}^\nu_{\mu\gamma}\partial_\sigma \mathbf{A}^\gamma] \\ & +[\Gamma^\nu_{\sigma\beta}\partial_\mu A^\beta + \Gamma^\nu_{\sigma\beta}\Gamma^\beta_{\mu\gamma}A^\gamma] \\ & -[\Gamma^\beta_{\sigma\mu}\partial_\beta A^\nu + \Gamma^\beta_{\sigma\mu}\Gamma^\nu_{\beta\gamma}A^\gamma]\end{aligned}$$

similarily,

$$\begin{aligned}\nabla_\mu \nabla_\sigma \mathbf{A}^\nu = & [\partial_\mu \partial_\sigma \mathbf{A}^\nu + \partial_\mu \mathbf{\Gamma}^\nu_{\sigma\beta}\mathbf{A}^\beta + \mathbf{\Gamma}^\nu_{\sigma\beta}\partial_\mu \mathbf{A}^\beta] \\ & +[\Gamma^\nu_{\mu\alpha}\partial_\sigma A^\alpha + \Gamma^\nu_{\mu\alpha}\Gamma^\alpha_{\sigma\beta}A^\beta] \\ & -[\Gamma^\alpha_{\mu\sigma}\partial_\alpha A^\nu + \Gamma^\alpha_{\mu\sigma}\Gamma^\nu_{\alpha\gamma}A^\gamma]\end{aligned}$$

and riemann Tensor is defined as

$$\mathbf{R}^\nu_{\alpha\mu\sigma} = \nabla_\sigma \nabla_\mu \mathbf{A}^\nu - \nabla_\mu \nabla_\sigma \mathbf{A}^\nu$$

Hence subtracting eq 13 from 12 , we get riemann Tensor

#It is also noted that for a flat space time Reimann tensor must be zero.

Ricci Tensor is the contraction of $R^\mu_{\alpha\nu\beta}$ on the first and third indices.Other contractions would in principle also be possible: on the first and second, the first and fourth, etc. But because $R^\alpha_{\beta\mu\nu}$ is antisymmetric on α and β and on μ and ν, all these contractions either vanish identically or reduce to $\pm R_{\alpha\beta}$. Therefore the Ricci tensor is essentially the only contraction of the Riemann tensor. Ricci scalar is contraction of Ricci tensor

$$R := g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}g^{\alpha\beta}R_{\alpha\mu\beta\nu}$$

Einstein's equation

My argument begins with the realization that I would like to find an equation that supersedes the Poisson equation for the Newtonian potential(ϕ):

$$\nabla^2\phi = 4\pi G\rho$$

where ∇^2 is laplacian in space and ρ is mass density. The analogy for mass density in tensorial form is energy momentum tensor $T_{\mu\nu}$ and it should be directly proportioanl to laplacian of a metric tensor.

$$[\nabla^2 g]_{\mu\nu} \propto T_{\mu\nu}$$

But it should also be noted that the left hand side tensor $g_{\mu\nu}$ is not a sensible tensor and the equation is needed to be completely tensorial. So our best choice is to use Riemann tensor which we defined earlier and it also represents space time curvature. Hence it must be related to energy momentum tensor in a linear fashion.

$$R_{\mu\nu} = kT_{\mu\nu}$$

But we know that $\nabla^\mu T_{\mu\nu} = 0$ from conservation of energy but we know that $\nabla^\mu R_{\mu\nu} \neq 0$ as from Bianchi identity we have $\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R$.

So if we take the term as a whole then we get ,

$$\nabla_\mu T = 0$$

Hence we define a new tensor , Einstein Tensor defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$$

Now we are left to propose,

$$G_{\mu\nu} = kT_{\mu\nu}$$

By contractiong both sides we get

$$R = -kT$$

from navier stoke's equation we get $k = 8\pi G$. so finally we get

$$\boxed{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}}$$

Scharzschild Metric and Radius

By performing tansformation from'spherical co-ordiantes to polar co-ordinates i.e.,

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2$$

we get the scharzschild metric from einstein's field equation as

$$ds^2 = -(1 + 2\phi)dt^2 + \frac{1}{1 + 2\phi}(dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2)$$

Then in order to get rid of the coefficients on $r^2d\theta^2 + r^2sin^2\theta d\phi^2$, we must replace r with R, where $r^2(1+\phi) = R^2$. By a handy coincidence then if $\phi = -\frac{C}{r}$ then it turns out that $dr = dR(1 + \frac{1}{2}\phi - \frac{1}{2}\phi + O(\phi^2)) = dR(1 + O(\phi^2))$ so to linear order we arrive at this metric.

$$ds^2 \approx -(1 - \frac{2GM}{c^2R})c^2dt^2 + \frac{dr^2}{1 - 2GM/c^2R} + r^2d\theta^2 + r^2sin^2\theta d\phi^2$$

where ϕ is gravitational potenital in weak field limit which is $\phi = -\frac{GM}{R}$

From equation we can easily see that there will be co-ordinate singularity when the coefficients are set to zero.

$$1 - \frac{2GM}{c^2R} = 0$$

$$R = \frac{2GM}{c^2}$$

which is the scharzschild radius R_s

4 Cosmology

4.1 What is Cosmology?

Cosmology is a branch of astronomy that involves the origin and evolution of the universe, from the Big Bang to today and on into the future. It is the study of the universe, or cosmos, regarded as a whole. The universe is richly textured with structures on a vast range of scales. It deals with topics as Dark Matter, Origin of The Universe, String Theory and so on.

4.1.1 How did universe come into being?

To answer this question , we must define some parameters and effects, some of which I have already discussed in the mathematical tools and prerequisites needed section. The left overs are Doppler effect(Red Shift), Hubble Constant, Critical density, Hubble rate etc.

1. Doppler Effect: In simple language doppler effect is when wave energy like sound or radio waves travels from two objects, the wavelength can seem to be changed if one or both of them are moving. This is Red shifted when the object(stars) is moving away from the observer(us). The red shift relation is given as
$$z \equiv \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}$$
where z is redshift paramter, if $z < 0$, this quantity is called a blueshift. Present observations tells us that a vast majority of galaxies have $z > 0$, which means galaxies are receding away from us.

2. Hubble Rate: This quantity is defined as
$$H(t) \equiv \frac{da/dt}{a}$$
where a is scale factor and da/dt is the rate by which scale factor changes over time. For example we know that for flat and matter dominated universe a varies as $a \propto t^{2/3}$, so $H = \frac{2}{3t}$. Hence $H.t = \frac{2}{3}$ and now H_0 is kept reserved for the Hubble rate in the present(now), and is called as Hubble Constant.

3. Friedmann Equation: The equation is given as
$$H^2(t) = \frac{8\pi G}{3}\left[\rho(t) + \frac{\rho_{cr} - \rho_0}{a^2(t)}\right]$$
here G is gravitational constant, ρ is where $\rho(t)$ is the energy density in the universe as a function of time with ρ_0 the present value. The critical density
$$\rho_{cr} = \frac{3H_0^2}{8\pi G}$$

4. Fluid and Acceleration Equations: From the first law of thermodynamics,

$$dQ = dE + PdV$$

where dQ is the heat flow into or out of a region, dE is the change in internal energy, P is the pressure, and dV is the change in volume of the region. This equation was applied to a comoving volume filled with photons, but it applies equally well to a comoving volume filled with any sort of fluid. If the universe is perfectly homogeneous, then for any volume $dQ = 0$ that is, there is no bulk flow of heat. The equation reduces to the form

$$\dot{E} + P\dot{V} = 0$$

The internal energy of the sphere is

$$E(t) = V(t)\epsilon(t)$$

and the volume of the sphere is:

$$V(t) = \frac{4\pi}{3} r_s^3 a(t)^3$$

Calculating the rate of change of the sphere's volume and the rate of change of the sphere's internal energy, we can write the fluid equation which goes by:

$$\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0$$

Combining the Friedmann equation and the Fluid equation, we obtain the acceleration equation which is:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P)$$

4.2 Cosmological Constant

The cosmological constant (Λ) is the value of the energy density of the vacuum of space. It was originally introduced by Albert Einstein in 1917 to achieve a static universe, which was the accepted view at the time. In fact, it was by no means settled that galaxies besides our own actually existed. After all, the sky is full of faint fuzzy patches of light. It took some time to sort out that some of the faint fuzzy patches are glowing clouds of gas within our Galaxy and that some of them are galaxies in their own right, far beyond our own Galaxy. Thus, when Einstein questioned the expansion or contraction of the universe, he looked not at the motions of galaxies, but at the motions of stars within our Galaxy, some of which are moving toward us and the others away from us, with no evidence that the Galaxy is expanding or contracting. The incomplete evidence available to Einstein led him to the belief that the universe is static. The question on the possibility of a static universe filled with non-relativistic matter and nothing else was then posed, to which it was reasoned that such a universe could not exist. If the mass density of the universe is ρ, then the gravitational potentialφ is given by Poisson's equation:

$$\nabla^2\varphi = 4\pi G\rho$$

The gravitational acceleration a at any point in space is then found by taking the gradient of the potential:

$$\overrightarrow{a} = -\overrightarrow{\nabla}\varphi$$

Since the universe is static, $\overrightarrow{a}$ must be zero everywhere in space. Thus, the potential f must be constant in space. Then we get $\rho = 0$, which indicates the universe is totally empty. A matter-filled universe which is initially static will contract under gravity. A matter-filled universe which is initially expanding, then it will either expand forever or reach a maximum radius and then collapse. To surmount this problem, Einstein introduced the new term Λ and rewrote the Poisson's equation in the form:

$$\nabla^2\varphi + \Lambda = 4\pi G\rho$$

Λ has dimensionality $(time)^{-2}$. Thus, this equation allows the universe to be static if $\Lambda = 4\pi G\rho$. If the Friedmann equation is rederived from Einstein's field equation, with the Λ term added, it becomes:

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3c^3}\epsilon - \frac{\kappa c^2}{R_0^2 a(t)^2} + \frac{\Lambda}{3}$$

The fluid equation is unaffected by the presence of Λ, but the acceleration equation becomes:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P) + \frac{\Lambda}{3}$$

For the universe to remain static, both a and a must be equal to zero. If $a = 0$, the Friedmann equation reduces to

$$0 = -\frac{4\pi G}{3} + \frac{\Lambda}{3}$$

Einstein's static model therefore had to be positively curved ($\kappa = +1$) Although Einstein published the details of his static, positively curved, matter-filled model, it was an unstable equilibrium. The energy density of Λ remains unchanged, but the energy density of matter drops. Thus, the repulsive force is greater than the attractive force, and the universe expands further, in turn causing the matter density to drop further, which causes the expansion to accelerate and so forth.

After Hubble's 1929 discovery that all galaxies outside the Local Group (the group that contains the Milky Way Galaxy) are moving away from each other, implying an overall expanding universe, Einstein abandoned the cosmological constant. However, the same paper that caused Einstein to abandonthe cosmological constant caused other scientists to embrace it. To make up for the age of the universe, which was badly estimated by Hubble due to his underestimation of the distance to galaxies,cosmologists introduced Λ again. This term has henceforth been in and out of the equations. In order to give Λ a real physical meaning, we need to identify some component of the universe whose energy density Λ remains constant as the universe expands or contracts. Currently, the leading candidate for this component is the vacuum energy. In classical physics, the idea of a vacuum having energy is wrong. In quantum physics, however, a vacuum is not a sterile void. The Heisenberg uncertainty principle permits particle-antiparticle pairs to spontaneously appear and then annihilate in an otherwise empty vacuum. There is an energy density ϵ_{vac} associated with the virtual particle-antiparticle pairs. Unfortunately, computing the numerical value of ϵ_{vac} is an exercise in quantum field theory which has not yet been successfully completed.

5 Discussion

In this project I learnt a lot about General Relativity, Cosmology and Dark Matter. I came across various questions and topics for further researches. Some of them need rigorous mathematical tools and concepts. Questions like "What happens to an entangled pair when one of them falls into a black hole?" has risen curiosity in me and I look forward to answer all of those questions in future by learning various concepts.

References

[1] Spacetime and Geometry: An Introduction to General Relativity by Sean Carroll

[2] An Introduction to Cosmology by J. V. Narlikar

[3] A First Course in General Relativity by Bernard Schutz

[4] Modern Cosmology by Scott Dodelson